Lecture 1.

What is the problem extending our notion of length of an interval in $\mathbb{R}$ to a notion of "measure" of an arbitrary set $E \subseteq \mathbb{R}$ in the naive way?

Prop 1. There does not exist a function $\mu : \mathcal{P}(\mathbb{R}) = \{E \subseteq \mathbb{R}\} \to [0, \infty]$ w/ the following properties:

(i) If $\{E_k\}_{k=1}^{\infty}$ are disjoint subsets, then

countable

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

(ii) If $E' = x + E$ for some $x \in \mathbb{R}$, then

$$\mu(E) = \mu(E')$$

(iii) $\mu([0,1)) = 1$.

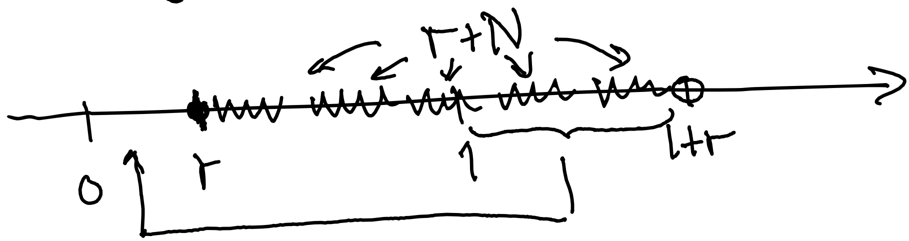
<u>Pf.</u> Suppose such μ exists. We shall derive a contradiction. Consider the equivalence relation on $[0,1)$: $x \sim y \iff x-y \in \mathbb{Q} = \left\{\begin{matrix}\text{rational}\\ \text{numbers}\end{matrix}\right\}$.

The equivalence classes are countable subsets of $[0,1)$ and their union $= [0,1)$.

By <u>Axiom of Choice</u>, $\exists\, N \subseteq [0,1)$ s.t.

N contains precisely one element from each equivalence class. For any $r \in \mathbb{Q} \cap [0,1)$ set

$$N_r = \{r+x : x \in N \cap [0,1-r)\} \cup \{r+x-1 : x \in N \cap [1-r,1)\}$$



By (i), (ii), $\boxed{\mu(N_r) = \mu(N)}$

We claim: ① $N_r \cap N_s = \emptyset$ if $r \neq s$

② $[0,1) = \bigcup_{r \in \mathbb{Q} \cap [0,1)} N_r$

To see ①, suppose $x \in N_r \cap N_s \Rightarrow$

$$\begin{cases} x = r+y \text{ or } r+y-1 & \text{for some } y \in N \\ x = s+z \text{ or } s+z-1 & \text{some } z \in N \end{cases}$$

Assume $x = r+y = s+z$ (other cases are similar).

Then $y - z = s - r \in \mathbb{Q} \Rightarrow y \sim z$. But since N contains only one point from each eq. class $\Rightarrow y = z \Rightarrow r = s. \Rightarrow$ ①.

To see ②, let $x \in [0,1)$ and let $y \in N$ s.t. $x \sim y$. Assume $x \geq y$ (other case similar).

Then $x - y = r \in \mathbb{Q} \cap [0,1) \Rightarrow x = y + r \Rightarrow$

$x \in N_r$

The claim now gives us a contradiction, since by props (i) and (ii) we must then have

$$1 = \mu([0,1)) = \sum_{\underbrace{r \in \mathbb{Q} \cap [0,1)}_{\text{countable}}} \mu(N_r).$$

This cannot be since $\mu(N_r) = \mu(N)$. $\boxtimes$

<u>Thm (Banach-Tarski)</u>. Let $U, V \subseteq \mathbb{R}^n$, $n \geq 3$. There exist $E_1, \dots, E_k$, $F_1, \dots, F_k$ s.t. E_j is <u>congruent</u> to F_j for $j = 1, \dots, k$,

differ by translation, rotation, reflection

$$E_i \cap E_j = F_i \cap F_j = \emptyset \text{ for } i \neq j,$$

and $U = \bigcup_{j=1}^{k} E_j$, $V = \bigcup_{j=1}^{k} F_j$.

This thm tells us that even if we relax condition (i) (in the higher dimensional analogue) to only require finite additivity:

$$\mu\left(\bigcup_{j=1}^{k} E_j\right) = \sum_{j=1}^{k} \mu(E_j), \quad E_j \cap E_l = \emptyset$$

we still cannot obtain such a measure.

The solution is to only require the measure to be defined on subcollection of sets in $P(\mathbb{R}^n)$. To produce a satisfactory theory, one assumes that the subcollection is a σ-algebra.